

The energy density of an Ising half plane lattice

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Abstract

We compute the energy density at arbitrary temperature of the half plane Ising lattice with a boundary magnetic field H_b at a distance M rows from the boundary and compare limiting cases of the exact expression with recent calculations at $T = T_c$ done by means of discrete complex analysis methods.

1 Introduction

Recently, Hongler and Smirnov [1] have studied the energy density of the Ising model at the critical temperature T_c on a half plane lattice a distance M rows from the boundary where two special cases of boundary conditions were considered 1) free and 2) fixed with all spins up. The problem was considered on an isotropic lattice with a boundary of arbitrary shape. When specialized to the half plane the results [2] are for free boundary conditions

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = -\frac{1}{2\pi M} + o(M^{-1}) \quad (1)$$

and for fixed plus spin boundary conditions

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = \frac{1}{2\pi M} + o(M^{-1}) \quad (2)$$

where in the isotropic lattice the vertical bulk energy density $\langle \mathcal{E}^v \rangle_{\text{bulk}}$ is

$$\langle \mathcal{E}^v \rangle_{\text{bulk}} \equiv \langle \sigma_{M,0} \sigma_{M-1,0} \rangle_{\text{bulk}} = \frac{1}{\sqrt{2}} \quad (3)$$

The computations of [1] are done by means of discrete complex analysis.

It is the purpose of this note to compute and study the energy density $\langle \sigma_{M,0} \sigma_{M-1,0} \rangle$ for the anisotropic lattice at arbitrary temperature on a half plane

with a magnetic field H_b applied to the boundary row. The energy operator is thus

$$\mathcal{E} = - \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} \{E_1 \sigma_{j,k} \sigma_{j,k+1} + E_2 \sigma_{j,k} \sigma_{j+1,k}\} - H_b \sum_{k=-\infty}^{\infty} \sigma_{1,k} \quad (4)$$

where we follow the notations of [3] and let $\sigma_{j,k}$ specify the spin in row j and column k . This reduces to the half plane case of [1] when $E_1 = E_2$ and $T = T_c$ with $H_b = 0$ for free boundary conditions and $H_b = \infty$ for plus spin boundary conditions. The exact result as calculated by Pfaffian methods is given in sec. 2. Limiting cases with $M \rightarrow \infty$ for $T < T_c$ and $T \rightarrow T_c^-$ are obtained in sec. 3 and for $T > T_c$ and $T \rightarrow T_c^+$ in sec. 4. We conclude in sec. 5 with a short discussion.

2 The energy density $\langle \sigma_{M,0} \sigma_{M-1,0} \rangle$ for arbitrary T and H_b

The computation of $\langle \sigma_{M,0} \sigma_{M-1,0} \rangle$ is done straightforwardly by means of Pfaffian methods. The details are given in chapter 7 of [3] and using the result (3.16d) on page 152 we immediately find that

$$\langle \sigma_{M-1,0} \sigma_{M,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = I \quad (5)$$

with

$$I = \frac{\alpha_2}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{1}{\alpha^{2M-1}} \frac{[(1 - z_1^2) - z_2 \alpha (1 + z_1^2 + 2z_1 \cos \theta)]}{[(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\alpha_2 \cos \theta)]^{1/2}} \times \left[\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz^2 z_2^{-1} v'/v} \right] \quad (6)$$

where we use the definitions

$$z_j = \tanh E_j / k_B T, \quad z = \tanh H_b / k_B T \quad (7)$$

$$\alpha_1 = \frac{z_1(1-z_2)}{(1+z_2)}, \quad \alpha_2 = \frac{(1-z_2)}{z_1(1+z_2)}, \quad (8)$$

the quantity v/v' is given by (3.7), (3.14) and (3.20) on pp 120–122 of [3], as

$$v/v' = \frac{z_2^2(1 + z_1^2 + 2z_1 \cos \theta) - z_2(1 - z_1^2)\alpha}{2z_1 z_2 \sin \theta} \quad (9)$$

α is the largest root of (3.2) on page 86 of [3]

$$(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \theta - z_2(1 - z_1^2)(\alpha + \alpha^{-1}) = 0 \quad (10)$$

which is explicitly given as

$$\alpha = \frac{1}{2z_2(1-z_1^2)} \left\{ (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \theta + z_1(1 + z_2)^2 [(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\alpha_2 \cos \theta)]^{1/2} \right\} \quad (11)$$

where the square root is defined to be positive for real θ . The vertical energy density $\langle \mathcal{E}^v \rangle_{\text{bulk}}$ is given by the equivalent forms

$$\begin{aligned} \langle \mathcal{E}^v \rangle_{\text{bulk}} &= z_2 + \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{(1 - z_1^2) - z_2 \alpha (1 + z_1^2 + 2z_1 \cos \theta)}{z_2 (1 - z_1^2) (1 - \alpha^2)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \left[\frac{(1 - \alpha_3 e^{i\theta})(1 - \alpha_4 e^{-i\theta})}{1 - \alpha_3 e^{-i\theta} (1 - \alpha_4 e^{i\theta})} \right]^{1/2} \end{aligned} \quad (12)$$

with

$$\alpha_3 = \frac{z_2(1 - z_1)}{1 + z_1}, \quad \alpha_4 = \frac{(1 - z_1)}{z_2(1 + z_1)} \quad (13)$$

where to obtain the last line of (12) we have used identities of [4].

We note in particular that when $H_b = 0$,

$$I = \frac{\alpha_2}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{1}{\alpha^{2M-1}} \frac{[(1 - z_1^2) - z_2 \alpha (1 + z_1^2 + 2z_1 \cos \theta)]}{[(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\alpha_2 \cos \theta)]^{1/2}} \quad (14)$$

3 Expansions for $M \rightarrow \infty$ for $T \leq T_c$

We make contact with the computations of [1] by computing the large M behavior of I as given by (6) in several cases.

3.1 $T < T_c$ and $H_b = 0$

When $H_b = 0$ and $T < T_c$ is fixed we obtain the large M behavior of I by expanding (14) by steepest descents. The maximum of the integrand of (14) is at $\theta = 0$ and thus expanding for small θ

$$\ln \alpha(\theta) \simeq \ln \left(\frac{z_2(1 + z_1)}{(1 - z_1)} \right) + \frac{z_1 \alpha_2}{(1 - \alpha_1)(1 - \alpha_2)} \theta^2, \quad (15)$$

$$\begin{aligned} &(1 - z_1^2) - z_2 \alpha (1 + z_1^2 + 2z_1 \cos \theta) \\ &= -\frac{1 + z_1}{1 - z_1} \{z_2^2(1 + z_1)^2 - (1 - z_1)^2\} + O(\theta^2), \end{aligned} \quad (16)$$

$$\begin{aligned} &\{(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\alpha_2 \cos \theta)\}^{1/2} \\ &= \frac{1}{z_1(1 + z_2)^2} \{z_2^2(1 + z_1)^2 - (1 - z_1)^2\} + O(\theta^2) \end{aligned} \quad (17)$$

we find

$$I \simeq -\frac{1}{2\pi} \frac{z_2(1 - z_2^2)(1 + z_1)^2}{(1 - z_1)^2} \int_{-\epsilon}^{\epsilon} d\theta e^{-2M \ln \alpha(\theta)} \quad (18)$$

Then in (18) set

$$u^2 = 2M\theta^2 \frac{z_1 \alpha_2}{(1 - \alpha_1)(1 - \alpha_2)} \quad (19)$$

and $\epsilon\sqrt{M} \rightarrow \infty$ we obtain the result that as $M \rightarrow \infty$

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = -\frac{z_2(1-z_2^2)(1+z_1)^2}{2(1-z_1)^2} \sqrt{\frac{(1-\alpha_1)(1-\alpha_2)}{2\pi z_1 \alpha_2 M}} \left[\frac{z_2(1+z_1)}{(1-z_1)} \right]^{-2M} \quad (20)$$

3.2 $T < T_c$ and $H_b > 0$

The large M behavior of I for $T < T_c$ and $H_b > 0$ fixed is obtained from (6) also by steepest descents where in addition to (15)-(17) we also need the expansion for $\theta \sim 0$

$$\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz^2 z_2^{-1} v'/v} \simeq \frac{z_2^2 A[(1-\alpha_2) - z^2 A]}{4z^2(1-\alpha_2)^2} \theta^2 \quad (21)$$

where

$$A = (1-\alpha_2) + \frac{\alpha_2(1+z_1)^2}{(1-\alpha_1)} = \frac{4}{(1+z_2)^2(1-\alpha_1)} \quad (22)$$

Thus we find as $M \rightarrow \infty$

$$\begin{aligned} & \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \\ &= -\frac{z_2^3(1-z_2)(1+z_1)^2}{8(1+z_2)(1-z_1)^2} \frac{[(1-\alpha_2) - z^2 A](1-\alpha_1)^{1/2}}{z^2 \sqrt{2\pi(1-\alpha_2)}(z_1 \alpha_2 M)^{3/2}} \left[\frac{z_2(1+z_1)}{(1-z_1)} \right]^{-2M} \end{aligned} \quad (23)$$

This is negative for

$$z^2 < (1-\alpha_2)/A = \frac{1}{4}(1+z_2)^2(1-\alpha_1)(1-\alpha_2) \quad (24)$$

and positive for

$$z^2 > \frac{1}{4}(1+z_2)^2(1-\alpha_1)(1-\alpha_2) \quad (25)$$

We note that both (20) and (23) have the same exponential decay but that (23) decays faster by a factor of $1/M$ than does (20). We also note as $z \rightarrow 0$ that the amplitude of (23) diverges. Therefore in order to connect together the regimes of $H_b = 0$ and $H_b > 0$ a crossover regime is required.

3.3 The crossover regime $T < T_c$, $H_b \rightarrow 0$ with $z^2 M$ fixed

The crossover from $H_b = 0$ to $H_b > 0$ for $T < T_c$ is obtained by considering $z \rightarrow 0$ and $M \rightarrow \infty$ with $z^2 M = O(1)$. Then when $M\theta^2 = O(1)$ we have the expansion which replaces (21)

$$\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz^2 z_2^{-1} v'/v} \sim \frac{M\theta^2}{M\theta^2 + 4z^2 M(1-\alpha_2)/z_2^2 A} \quad (26)$$

Then using (15)-(17),(19) and setting

$$\zeta^2 = \frac{8z^2 M z_1 \alpha_2}{(1 - \alpha_1) z_2^2 A} = 2M z^2 (1 + z_2)^2 \alpha_2 z_1 z_2^{-2} \quad (27)$$

we find from (6) that the crossover function is

$$\begin{aligned} & \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \\ &= -\frac{z_2(1 - z_2^2)(1 + z_1)^2}{2\pi(1 - z_1)^2} \left(\frac{(1 - \alpha_1)(1 - \alpha_2)}{2z_1 \alpha_2 M} \right)^{1/2} \left(\frac{z_2(1 + z_1)}{1 - z_1} \right)^{-2M} \int_{-\infty}^{\infty} du \frac{u^2}{u^2 + \zeta^2} e^{-u^2} \end{aligned} \quad (28)$$

When $\zeta \rightarrow 0$ (28) reduces to (20) and when $\zeta \rightarrow \infty$ (28) reduces to (23) with $z \rightarrow 0$.

3.4 $T \rightarrow T_c$ — and $H_b = 0$

In order to obtain the results (1) and (2) of [1] at $T = T_c$ where

$$z_{1c} z_{2c} + z_{1c} + z_{2c} = 1 \quad \alpha_2 = 1 \quad (29)$$

we consider $\alpha_2 \rightarrow 1$ in the asymptotic expansions for $M \rightarrow \infty$ (20),(23),(28) and observe that the exponential factor

$$\frac{z_2(1 + z_1)}{1 - z_1} \sim e^{4z_{1c}(1 - \alpha_2)/(1 - z_{1c})} \quad (30)$$

and that the coefficients either diverge or vanish. Therefore when $1 - \alpha_2 \rightarrow 0$ a separate expansion is needed. In the integral (6) for I we set

$$\theta/(1 - \alpha_2) = x \quad (31)$$

with $\theta \rightarrow 0$ and $\alpha_2 \rightarrow 1$ with x fixed of order one and use the approximations

$$\alpha \sim 1 + \frac{2z_{1c}}{1 - z_{1c}^2} (1 - \alpha_2) \sqrt{1 + x^2} \quad (32)$$

and

$$\frac{\alpha_2 \alpha [(1 - z_1^2) - z_2 \alpha (1 + z_1^2 + 2z_1 \cos \theta)]}{[(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\cos \theta)]^{1/2}} \sim -\frac{2z_{1c}}{1 - z_{1c}^2} \left[1 + \frac{1}{\sqrt{1 + x^2}} \right]. \quad (33)$$

Then, defining for $M \rightarrow \infty$ and $1 - \alpha_2 \rightarrow 0$

$$m = \frac{4z_{1c}}{1 - z_{1c}^2} M (1 - \alpha_2) \quad (34)$$

we obtain the result

$$\begin{aligned} & \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \\ &= -\frac{z_{1c}(1 - \alpha_2)}{\pi(1 - z_{1c}^2)} \int_{-\infty}^{\infty} dx \left[1 + \frac{1}{\sqrt{1 + x^2}} \right] e^{-m\sqrt{1 + x^2}} \end{aligned} \quad (35)$$

which, using $x = \sinh y$ is rewritten as

$$\begin{aligned} \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} &= -\frac{z_{1c}(1-\alpha_2)}{\pi(1-z_{1c}^2)} \int_{-\infty}^{\infty} dy [\cosh y + 1] e^{-m \cosh y} \\ &= -\frac{2z_1(1-\alpha_2)}{\pi(1-z_1^2)} [K_1(m) + K_0(m)] \end{aligned} \quad (36)$$

where $K_n(z)$ is the modified Bessel function of order n [5].

When $m \rightarrow \infty$ we use the first term in the expansion

$$K_n(m) = \sqrt{\frac{\pi}{2m}} e^{-m} \left(1 + \frac{4n^2 - 1}{8m} + O(m^{-2}) \right) \quad (37)$$

to find that (36) reduces to

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = -\sqrt{\frac{2z_{1c}(1-\alpha_2)}{\pi(1-z_{1c}^2)M}} e^{-4z_{1c}M(1-\alpha_2)/(1-z_{1c}^2)} \quad (38)$$

which agrees with (20) in the limit $\alpha_2 \rightarrow 1$ when we use (30).

When $m \rightarrow 0$ we use

$$K_1(m) \sim 1/m, \quad K_0(m) \sim -\ln m \quad (39)$$

to find that (36) reduces to

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = -\frac{1}{2\pi M} \quad (40)$$

which agrees with the result of [1] for free boundary conditions (1).

3.5 $T \rightarrow T_c-$ and $H_b > 0$

When $T \rightarrow T_c$ with $H_b > 0$ we need the further approximation that by using (31) in (9)

$$v/v' \sim \frac{1 - \sqrt{1+x^2}}{x} \quad (41)$$

and thus

$$\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz^2 z_2^{-1} v'/v} \sim - \left[\frac{\sqrt{1+x^2} - 1}{x} \right]^2 \quad (42)$$

Using (42) in (6) with (31)-(34) we obtain

$$I \sim \frac{z_{1c}(1-\alpha_2)}{\pi(1-z_{1c}^2)} \int_{-\infty}^{\infty} \left[\frac{\sqrt{1+x^2} - 1}{x} \right]^2 \left[1 + \frac{1}{\sqrt{1+x^2}} \right] e^{-m\sqrt{1+x^2}} \quad (43)$$

which setting $x = \sinh y$ gives the result

$$\begin{aligned} \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} &= \frac{z_{1c}(1-\alpha_2)}{\pi(1-z_{1c}^2)} \int_{-\infty}^{\infty} dy (\cosh y - 1) e^{-m \cosh y} \\ &= \frac{2z_{1c}(1-\alpha_2)}{\pi(1-z_{1c}^2)} [K_1(m) - K_0(m)] \end{aligned} \quad (44)$$

which differs from (36) only in the sign of the term with $K_1(m)$.

When $m \rightarrow \infty$ we use (37) in (44) to find

$$\begin{aligned} \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} &= \\ \frac{1}{8} \sqrt{\frac{1-z_{1c}^2}{2\pi z_{1c}(1-\alpha_2)}} \frac{1}{M^{3/2}} e^{-4z_{1c}M(1-\alpha_2)/(1-z_{1c}^2)} \end{aligned} \quad (45)$$

which agrees with (23) with $\alpha_2 \rightarrow 1$.

When $m \rightarrow 0$ we use (39) to find

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = \frac{1}{2\pi M} \quad (46)$$

which agrees with the result of [1] for fixed spin boundary conditions (2).

3.6 The crossover regime $T \rightarrow T_c-$, $H_b \rightarrow 0$ with $z^2 M$ fixed

It is of further interest to determine the crossover between the results (36) and (44) and the specialization to the crossover between the two results (1) and (2). When $z \rightarrow 0$ and $M \rightarrow \infty$ with $z^2 M = O(1)$ and when $M(1-\alpha_2) = O(1)$, we have the expansion which replaces (42)

$$\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz^2 z_2^{-1} v'/v} \sim \frac{mx^2[\sqrt{1+x^2} - 1] - \zeta_c^2[\sqrt{1+x^2} - 1]^2}{mx^2[\sqrt{1+x^2} - 1] + \zeta_c^2 x^2} \quad (47)$$

where ζ_c^2 is obtained from (27) with $\alpha_2 \rightarrow 1$ as

$$\zeta_c^2 = 2z^2 M(z_{2c}^{-2} - 1) \quad (48)$$

Then using (47) in (6) with (31)-(34) we obtain,

$$I \sim -\frac{z_{1c}(1-\alpha_2)}{\pi(1-z_{1c}^2)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{1+x^2}} \frac{mx^2 - \zeta_c^2[\sqrt{1+x^2} - 1]}{m[\sqrt{1+x^2} - 1] + \zeta_c^2} e^{-m\sqrt{1+x^2}} \quad (49)$$

which setting $x = \sinh y$ gives the result,

$$\begin{aligned} \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} &= \\ -\frac{z_{1c}(1-\alpha_2)}{\pi(1-z_{1c}^2)} \int_{-\infty}^{\infty} dy \frac{m(\cosh y + 1) - \zeta_c^2}{m(\cosh y - 1) + \zeta_c^2} (\cosh y - 1) e^{-m \cosh y} \end{aligned} \quad (50)$$

When $\zeta_c^2 \rightarrow 0$, (36) is recovered, and when $\zeta_c^2 \rightarrow \infty$ (44) is recovered.

Finally in order to interpolate between the results (1) and (2) we let $m \rightarrow 0$ in (49) and set $mx = t$ to obtain

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = -\frac{1}{2\pi M} \int_0^\infty dt \frac{t - \zeta_c^2}{t + \zeta_c^2} e^{-t} \quad (51)$$

In Figure 1, the integral in (51) is evaluated numerically for various ζ_c^2 . The integral vanishes at $\zeta_c^2 = 0.610058 \dots$.

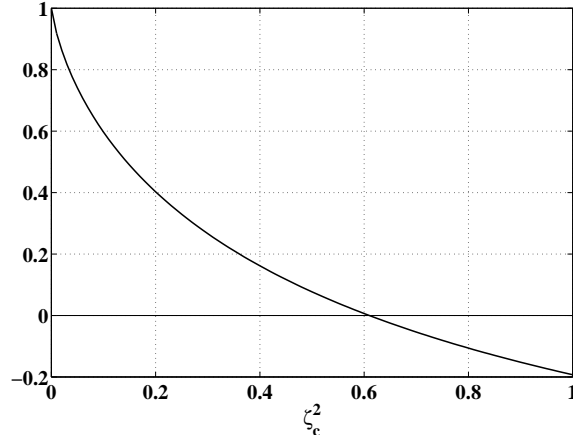


Figure 1: Numerical plot of the integral in (51).

4 Expansions for $M \rightarrow \infty$ for $T > T_c$

The fundamental result (6) holds for $T > T_c$ as well as $T < T_c$. The analysis of the various special limiting cases is parallel to $T < T_c$ where now $\alpha_2 > 1$ and (15) – (17) are replaced by

$$\ln \alpha(\theta) \sim \ln \left(\frac{(1 - z_1)}{z_2(1 + z_1)} \right) + \frac{z_1 \alpha_2}{(1 - \alpha_1)(\alpha_2 - 1)} \theta^2 \quad (52)$$

$$\begin{aligned} & (1 - z_1^2) - z_2 \alpha (1 + z_1^2 + 2z_1 \cos \theta) \\ & \sim -\frac{4z_1(1 - z_1)}{(1 + z_2)^2(1 + z_1)(1 - \alpha_1)(\alpha_2 - 1)} \theta^2 \end{aligned} \quad (53)$$

$$\begin{aligned} & \{(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\alpha_2 \cos \theta)\}^{1/2} \\ & = (1 - \alpha_1)(\alpha_2 - 1) + O(\theta^2) \end{aligned} \quad (54)$$

4.1 $T > T_c$ and $H_b = 0$

Using (52)–(54) in (6) we find that $M \rightarrow \infty$,

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}}$$

$$= -\frac{(1-z_1)^2}{2\alpha_2(1+z_1)^2(1+z_2)^2\sqrt{2\pi z_1\alpha_2(1-\alpha_1)(\alpha_2-1)}M^{3/2}} \left[\frac{(1-z_1)}{z_2(1+z_1)} \right]^{-2M} \quad (55)$$

which is to be compared with the corresponding result (20) for $T < T_c$.

4.2 $T > T_c$ and $H_b > 0$

When $T > T_c$, and $\theta \sim 0$

$$v/v' \sim -\frac{(1+z_2)^2(1-\alpha_1)(\alpha_2-1)}{2z_1\theta} \quad (56)$$

and (21) is replaced by,

$$\frac{(e^{i\theta}-1)/(e^{i\theta}+1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta}-1)/(e^{i\theta}+1) - iz^2 z_2^{-1} v'/v} \simeq -\frac{z^2(1+z_2)^2(1-\alpha_1)(\alpha_2-1)^2}{z_2^2[(\alpha_2-1) + z^2 A]\theta^2} \quad (57)$$

Thus as $M \rightarrow \infty$

$$\begin{aligned} & \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \\ &= \frac{2(1-z_1)^2 z^2}{z_2^2(1+z_1)^2[(\alpha_2-1) + z^2 A]} \sqrt{\frac{z_1(\alpha_2-1)}{2\pi M \alpha_2(1-\alpha_1)}} \left[\frac{(1-z_1)}{z_2(1+z_1)} \right]^{-2M} \end{aligned} \quad (58)$$

with A given by (22). The result (58) is positive for all $z^2 > 0$ in contrast with the corresponding result (23) for $T < T_c$ which changes sign at $z^2 = (1-\alpha_2)/A$.

4.3 The crossover regime $T > T_c$, $H_b \rightarrow 0$ with $z^2 M$ fixed

In this case we find that

$$\frac{(e^{i\theta}-1)/(e^{i\theta}+1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta}-1)/(e^{i\theta}+1) - iz^2 z_2^{-1} v'/v} \simeq 1 - \zeta^2/\bar{u}^2 \quad (59)$$

with ζ^2 defined by (27) and

$$\bar{u}^2 = 2M\theta^2 \frac{z_1\alpha_2}{(1-\alpha_1)(\alpha_2-1)} \quad (60)$$

thus we obtain the result

$$\begin{aligned} & \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \\ &= -\frac{(1-z_1)^2(1-2\zeta^2)}{2\alpha_2(1+z_1)^2(1+z_2)^2\sqrt{2\pi z_1\alpha_2(1-\alpha_1)(\alpha_2-1)}M^{3/2}} \left[\frac{(1-z_1)}{z_2(1+z_1)} \right]^{-2M} \end{aligned} \quad (61)$$

which agrees with (55) when $\zeta \rightarrow 0$ and agrees with the $z \rightarrow 0$ limit of (58) when $\zeta \rightarrow \infty$.

4.4 $T \rightarrow T_c +$ and $H_b = 0$

Approaching T_c from above, (32) is replaced by,

$$\alpha \sim 1 + \frac{2z_{1c}}{1 - z_{1c}^2}(\alpha_2 - 1)\sqrt{1 + x^2} \quad (62)$$

where now,

$$x = \theta/(\alpha_2 - 1) \quad (63)$$

with $\theta \rightarrow 0$ and $\alpha_2 \rightarrow 1$ with x fixed of order one and (33) is replaced by

$$\frac{\alpha_2 \alpha [(1 - z_1^2) - z_2 \alpha (1 + z_1^2 + 2z_1 \cos \theta)]}{[(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\cos \theta)]^{1/2}} \sim -\frac{2z_{1c}}{1 - z_{1c}^2} \left[1 - \frac{1}{\sqrt{1 + x^2}} \right]. \quad (64)$$

Defining

$$\bar{m} = \frac{4z_{1c}}{1 - z_{1c}^2} M(\alpha_2 - 1) \quad (65)$$

and setting $x = \sinh y$ we find

$$\begin{aligned} \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} &= -\frac{z_{1c}(\alpha_2 - 1)}{\pi(1 - z_{1c}^2)} \int_{-\infty}^{\infty} dy [-1 + \cosh y] e^{-\bar{m} \cosh y} \\ &= -\frac{2z_{1c}(\alpha_2 - 1)}{\pi(1 - z_{1c}^2)} [-K_0(\bar{m}) + K_1(\bar{m})] \end{aligned} \quad (66)$$

which is to be compared with (36).

When $\bar{m} \rightarrow \infty$, (66) reduces to

$$\begin{aligned} \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} &= \\ &= -\frac{1}{8} \sqrt{\frac{1 - z_{1c}^2}{2\pi z_{1c}(\alpha_2 - 1)}} \frac{1}{M^{3/2}} e^{-4z_{1c}M(\alpha_2 - 1)/(1 - z_{1c}^2)} \end{aligned} \quad (67)$$

which agrees with (55) with $\alpha_2 \rightarrow 1$.

When $\bar{m} \rightarrow 0$, (66) reduces to the result (1)

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = -\frac{1}{2\pi M} \quad (68)$$

4.5 $T \rightarrow T_c +$ and $H_b > 0$

When $T \rightarrow T_c$ from above, (41) is replaced by

$$\frac{v}{v'} \sim -\frac{\sqrt{1 + x^2} + 1}{x} \quad (69)$$

and (42) is replaced by

$$\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz^2 z_2^{-1} v'/v} \sim -\left[\frac{\sqrt{1 + x^2} + 1}{x} \right]^2 \quad (70)$$

Thus setting $x = \sinh y$ we obtain the result

$$\begin{aligned} \langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} &= \frac{z_{1c}(\alpha_2 - 1)}{\pi(1 - z_{1c}^2)} \int_{-\infty}^{\infty} dy (\cosh y + 1) e^{-\bar{m} \cosh y} \\ &= \frac{2z_{1c}(\alpha_2 - 1)}{\pi(1 - z_{1c}^2)} [K_1(\bar{m}) + K_0(\bar{m})] \end{aligned} \quad (71)$$

which is to be compared with (44).

When $\bar{m} \rightarrow \infty$, (71) reduces to

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = \sqrt{\frac{2z_{1c}(\alpha_2 - 1)}{\pi(1 - z_{1c}^2)M}} e^{-4z_{1c}M(\alpha_2 - 1)/(1 - z_{1c}^2)} \quad (72)$$

which agrees with (58) with $\alpha_2 \rightarrow 1$.

When $\bar{m} \rightarrow 0$, (71) reduces to

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} = \frac{1}{2\pi M} \quad (73)$$

which agrees with the result (2).

4.6 The crossover regime $T \rightarrow T_c+$, $H_b \rightarrow 0$ with $z^2 M$ fixed

In this case we have

$$\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz^2 z_2^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz^2 z_2^{-1} v'/v} \sim \frac{\bar{m} x^2 [\sqrt{1 + x^2} + 1] - \zeta_c^2 [\sqrt{1 + x^2} + 1]^2}{\bar{m} x^2 [\sqrt{1 + x^2} + 1] + \zeta_c^2 x^2} \quad (74)$$

Thus, using (64) and setting $x = \sinh y$, we obtain

$$\begin{aligned} &\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \\ &= -\frac{z_{1c}(\alpha_2 - 1)}{\pi(1 - z_{1c}^2)} \int_{-\infty}^{\infty} dy \frac{\bar{m}(\cosh y - 1) - \zeta_c^2}{\bar{m}(\cosh y + 1) + \zeta_c^2} (\cosh y + 1) e^{-\bar{m} \cosh y} \end{aligned} \quad (75)$$

which is to be compared with (50). When $\zeta_c^2 \rightarrow 0$, (66) is recovered, and when $\zeta_c^2 \rightarrow \infty$ (71) is recovered. When $\bar{m} \rightarrow 0$ the result (51) is again obtained.

5 Discussion

In this paper we have derived leading behavior of the energy density operator of the Ising model on an anisotropic lattice M rows from a half plane boundary at critically with a magnetic field H_b on the boundary by use of Pfaffian methods. When the field is zero and infinity we regain the results (1) and (2) obtained in [1] by means of discrete complex analysis. Furthermore in (51) we have obtained the result in the more general situation where $H_b^2 M$ is fixed with $H_b \rightarrow 0$ and $M \rightarrow \infty$. This result goes beyond the computations of [1] and we have obtained

many results for $T \neq T_c$. It is of interest to obtain these results also by the methods of discrete complex analysis.

We would also like to take this opportunity to remark that it would be most useful to extend the results of [1] in several directions. One such direction is to consider discrete complex analysis on surfaces of higher genus and to derive and extend the results of [6] and [7].

A second direction is to consider inhomogeneous random lattices. One such case is the layered Ising model where the vertical interaction constants are the same in all columns but are chosen randomly from row to row [3, chapters 14 and 15] where it is known that there is an entire temperature region around T_c where the correlation functions are algebraic. All of these problems can be considered as problems with free fermions and thus the methods of discrete complex analysis should apply.

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